Mathematical Preliminaries

1 Sets

- A set is an unordered collection of distinct objects (elements). These elements may be anything, including other sets.

- Special sets include:
  - \( N \) - set of natural numbers (integers beginning with 0).
  - \( R \) or \( \mathbb{R} \) - set of real numbers.
  - \( \{\} \) - null or empty set.

- The elements of a set may be specified in a number of ways, the simplest being:
  \[ V = \{a, e, u, o, i\} \]
  
  The notation \( a \in V \) means “\( a \) is an element of \( V \).”

  Another way to specify a set is to describe the elements logically in “set-builder notation”:
  \[ S = \{x | x \in \mathbb{R} \text{ and } 0 \leq x \leq 1\} \]
  
  is the set of all real numbers in the interval from 0 to 1 inclusive.

- \( S \) is a subset of \( T \) if and only if (iff) each element of \( S \) is also an element of \( T \), and is denoted by \( S \subseteq T \).

- The power set of \( S \), denoted by \( P(S) \), is the collection of all subsets of \( S \).

- The cardinality of a set \( S \), denoted by \( |S| \), is the number of distinct elements of \( S \).

- New sets may be formed from existing sets \( S \) and \( T \):
  - union: \( S \cup T \).
  - intersection: \( S \cap T \).
  - difference: \( S - T \).
  - complement: \( \overline{S} \).

- Two sets, \( A \) and \( B \), are disjoint if \( A \cap B = \{\} \).
2 Functions

- A function \( f \) is a mapping from one set, \( A \), called the domain, to another set \( B \), called the co-domain, that associates each element of \( A \) with a single element of \( B \):
  
  If \( a \in A \) and \( f \) maps \( a \) to \( b \), we write \( f(a) = b \).

- The range of \( f \) is the set containing all \( b \in B \) such that \( f(a) = b \) for some \( a \in A \).

- Special functions of interest:
  
  - Floor: \( f(x) = \lfloor x \rfloor \). This is the largest integer less than or equal to \( x \).
  - Ceiling: \( f(x) = \lceil x \rceil \). This is the smallest integer greater than or equal to \( x \).
  - Factorial: \( f(n) = n! \). Note that the domain of this function is the natural numbers.

- Exponentiation (base \( b \)): \( f(x) = b^x \). Rules for manipulation:

  \[
  b^x b^y = b^{x+y} \tag{1}
  \]
  \[
  \frac{b^x}{b^y} = b^{x-y} \tag{2}
  \]
  \[
  (b^x)^y = b^{x y} \neq b^{(x^y)} \tag{3}
  \]

- Logarithm \((b > 0)\): \( f(x) = \log_b x \). This means \( y = \log_b x \) if \( b^y = x \). If \( b \) is unspecified, 2 is assumed.
  
  - Some common notation:

    \[
    \lg n = \log_2 n \tag{4}
    \]
    \[
    \lg n^2 = \log_2 n^2 = \log_2(n^2) \tag{5}
    \]
    \[
    \lg^2 n = \log_2^2 n = (\log_2 n)^2 \tag{6}
    \]

  - Rules for manipulation:

    \[
    \log_b 1 = 0 \tag{7}
    \]
    \[
    \log_b b = 1 \tag{8}
    \]
    \[
    \log_a x = \frac{\log_b x}{\log_b a} \text{ where } b > 0 \tag{9}
    \]
    \[
    \log x y = \log x + \log y \tag{10}
    \]
    \[
    \log x^y = y \log x \neq (\log x)^y \tag{11}
    \]
    \[
    \log x < x \text{ for all } x > 0 \tag{12}
    \]
    \[
    \frac{d}{dx} \log_b x = \frac{1}{x \ln b} \tag{13}
    \]
3 Summations

- A summation is a short-hand notation to describe the addition of the terms of a sequence:

\[ \sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \ldots + f(n) \]

The variable \( i \) is called the index variable. Usually \( m = 0 \) or \( m = 1 \).

- Special summation formulas:

\[ \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \]  \hspace{1cm} (14)
\[ \sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1} \quad \text{where} \ a \neq 1 \]  \hspace{1cm} (15)
\[ \sum_{i=0}^{n} i = \frac{n(n+1)}{2} \]  \hspace{1cm} (16)
\[ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \]  \hspace{1cm} (17)
\[ \sum_{i=1}^{n} \frac{1}{i} \approx \log_e n \]  \hspace{1cm} (18)

- Some rules for manipulating summations, where \( c \) is a constant:

\[ \sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i \]  \hspace{1cm} (19)
\[ \sum_{i=m}^{n} (a_i \pm b_i) = \sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i \]  \hspace{1cm} (20)
\[ \sum_{i=m}^{n} c = c(n - m + 1) \]  \hspace{1cm} (21)
\[ \sum_{i=m}^{n} a_i = \sum_{i=0}^{m-1} a_i - \sum_{i=0}^{n-m} a_i \]  \hspace{1cm} (22)
\[ \sum_{i=m}^{n} a_i = \sum_{j=0}^{n-m} a(j + m) \]  \hspace{1cm} (23)